

# Exact coherent states in one-dimensional quantum many-body systems with inverse-square interactions

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For the models of  $N$ -body identical harmonic oscillators interacting through potentials of homogeneous degree -2, the unitary operator that transforms a system of time-dependent parameters into that of unit spring constant and unit mass of different timescale is found. If the interactions can be written in terms of the differences between positions of two particles, it is also shown that the Schrödinger equation is invariant under a unitary transformation. These unitary relations can be used not only in finding coherent states from the given stationary states in a system, but also in finding exact wave functions of the Hamiltonian systems of time-dependent parameters from those of time-independent Hamiltonian systems. Both operators are invariant under the exchange of any pair of particles. The transformations are explicitly applied for some of the Calogero-Sutherland models to find exact coherent states.

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## I. INTRODUCTION

The harmonic oscillator (of time-dependent parameter) is a model where the path integral for the kernel (propagator) is Gaussian and thus the kernel can be almost determined by the classical action. From the fact that the kernel should satisfy Schrödinger equation, with a condition for the kernel in the coincident limit, one can obtain the exact expression of the kernel in terms of the solutions of classical equation of motion [1]. This is one of the basic reasons of the fact that wave functions of harmonic oscillators are described by the solutions of

classical equation of motion. Two different types of operators have been known for the construction of coherent states from the vacuum state of a simple harmonic oscillator system; the displacement operator and squeeze operator [2–4]. Recently, it has been shown [5] that a harmonic oscillator of time-dependent parameters are unitarily equivalent to a simple harmonic oscillator of different timescale (including the case with an inverse-square potential [6]), where the unitary operator of the relation is again given in terms of the classical solutions of the system of time-dependent parameters. If one considers

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the set of wave functions whose centers of the probability distribution functions do not move, the operator at a given time corresponds to a squeeze operator. As have been well known, a driven harmonic oscillator (without an inverse-square potential) is unitarily equivalent to a harmonic oscillator without driving force, and this relation can be used in finding the unitary transformation which does not change the form of the Schrödinger equation in a harmonic oscillator. The unitary operator for this transformation corresponds to a displacement operator. By applying this transformation to the eigenstates of stationary probability distribution of the model, one can obtain the wave functions whose centers of the probability distributions move according to the classical solution.

In this paper, we will consider the models of identical  $N$ -body harmonic oscillators of time-dependent parameters interacting through the potential  $V(x_1, x_2, \dots, x_N)$  of homogeneous degree -2 which satisfies

$$V(ax_1, ax_2, \dots, ax_N) = a^{-2}V(x_1, x_2, \dots, x_N) \quad (1)$$

with non-zero constant  $a$ . One of the models which belongs to this category was first solved by Sutherland [7], based on the earlier work by Calogero [8]. From its inception, this model is closely related to the random matrix model [9,10] and has been found relevant for the descriptions of various physical phenomena [10]. The model has been generalized into the several cases [11–13], known in the literature as the Calogero-Sutherland models, which have generated wide interest [14,15], while some of the

generalized models do not belong to the category we will consider (see Sec. V.).

We will show that, there is a unitary transformation which relates the system to the the model of different timescale with unit mass and spring constant. The operator for the transformation is given as a product of the squeeze-type unitary operators in (one-body) harmonic oscillators. If  $V(x_1, x_2, \dots, x_N)$  is written in terms of the differences between positions of two particles, so that

$$V(x_1 + a, x_2 + a, \dots, x_N + a) = V(x_1, x_2, \dots, x_N), \quad (2)$$

there also exists a unitary transformation which does not change the form of the Schrödinger equation, while the operator for the transformation is given as the product of the displacement-type unitary operators in (one-body) harmonic oscillators. Both unitary operators are symmetric under the exchange of any pair of particles. These operators, as in the one-body harmonic oscillator case [5], thus can be used to find the wave functions of the systems of time-dependent parameters from those of constant parameters, preserving the symmetric property of the wave functions. The unitary transformations will be explicitly applied for three cases in the Calogero-Sutherland models, to find exact wave functions. In a Sutherland model of time-dependent parameters, Sutherland [16] has found an exact "coherent" state by directly analyzing the Schrödinger equation; we will show that this "coherent" state is obtained by applying the squeeze-type operator, while the displacement-type operator can also be applied

in this model to give a more general form of the exact wave function. In addition, by extending the definition of the displacement-type operator, we will also show that one can find a unitary relation between the system without external force and the same system with external force. Since the quantum states we will find through unitary transformations can be solely described, up to some parameters of the models, by the classical solutions of a harmonic oscillator, we will call them coherent states.

This paper will be organized as follows; in the next section, we will introduce the squeeze-type unitary operator which relates the interacting  $N$ -body oscillator system of time-dependent parameters and the same system of constant parameters. The displacement-type operator will be also introduced, and it will be shown that, if the potential of interaction satisfies Eq. (2), the unitary transformation by the operator does not change the form of the Schrödinger equation. In Sec. III, the unitary transformations will be explicitly applied for three cases in the Calogero-Sutherland models. In Sec. IV, for the cases that the potential of the interaction satisfies Eq. (2), it will be shown that the displacement-type operator can be extended to give the unitary relation between a system without external force and the same system with external force. The last section will be devoted to a summary and discussions.

## II. THE UNITARY TRANSFORMATIONS

It has been shown that [5], the harmonic oscillator with inverse-square potential described by the Hamiltonian

$$H_{s,1} = \frac{p^2}{2} + \frac{x^2}{2} + \frac{g}{x^2} \quad (3)$$

with a coupling constant  $g$ , and the oscillator with time-dependent mass  $M(t)$ , spring constant  $w(t)$  described by the Hamiltonian

$$H_1 = \frac{p^2}{2M(t)} + \frac{1}{2}M(t)w^2(t)x^2 + \frac{g}{M(t)}\frac{1}{x^2} \quad (4)$$

are related through the unitary transformation. For the case of  $g = 0$ , the classical equation of motion for the system of the Hamiltonian in Eq. (4) is written as;

$$\frac{d}{dt}(M\dot{x}) + M(t)w^2(t)x = 0. \quad (5)$$

If we denote the two linearly independent solutions of Eq. (5) as  $u(t)$  and  $v(t)$ , the  $\rho(t)$  defined by  $\rho(t) = \sqrt{u^2 + v^2}$  satisfies

$$\frac{d}{dt}(M\dot{\rho}) - \frac{\Omega^2}{M\rho^3} + Mw^2\rho = 0 \quad (6)$$

with a time-constant  $\Omega \equiv M(t)[\dot{v}(t)u(t) - \dot{u}(t)v(t)]$ , while the overdots denote differentiation with respect to  $t$ . Without losing generality we assume that  $\Omega$  is positive.

By defining the operator  $O_{s,1}$  and  $O_1$  as

$$\begin{aligned} O_{s,1}(\tau) &= -i\hbar \frac{\partial}{\partial \tau} + H_{s,1} \\ O_1(t) &= -i\hbar \frac{\partial}{\partial t} + H_1, \end{aligned} \quad (7)$$

if the  $\tau$ , the time of the system of  $H_{s,1}$ , and  $t$ , the time of the system of  $H_1$ , is related as

$$d\tau = \frac{\Omega}{\rho^2} dt, \quad (8)$$

the unitary relation between the two operators has been given in Ref. [5] as

$$U_1 O_{s,1}(\tau) U_1^\dagger |_{\tau=\tau(t)} = \frac{M\rho^2}{\Omega} O_1, \quad (9)$$

where

$$U_1 = \exp\left[\frac{i}{2\hbar} M \frac{\dot{\rho}}{\rho} x^2\right] \exp\left[-\frac{i}{4\hbar} \ln\left(\frac{\rho^2}{\Omega}\right) (xp + px)\right]. \quad (10)$$

There is a similar unitary relation between the identical  $N$ -body harmonic oscillators interacting through the potential  $V(x_1, x_2, \dots, x_N)$  satisfying Eq. (1). The potential  $V$  may be written as a linear combination of the terms  $1/\sum_{l,m=1}^N a_{lm} x_l x_m$ , where  $a_{lm} = a_{ml}$ . If a system is described by the Hamiltonian

$$H_{s,N} = \sum_{i=1}^N \left( \frac{p_i^2}{2} + \frac{x_i^2}{2} \right) + V(x_1, x_2, \dots, x_N) \quad (11)$$

and another system is described by the Hamiltonian

$$H_N = \sum_{i=1}^N \left( \frac{p_i^2}{2M(t)} + M(t) w^2(t) \frac{x_i^2}{2} \right) + \frac{1}{M(t)} V(x_1, x_2, \dots, x_N), \quad (12)$$

from the commutator relation

$$\begin{aligned} & \left[ \sum_i (x_i p_i + p_i x_i), 1/\sum_{l,m=1}^N a_{lm} x_l x_m \right] \\ &= 4i\hbar / \sum_{l,m=1}^N a_{lm} x_l x_m, \end{aligned} \quad (13)$$

one may find the unitary relation of the two systems

$$U_N O_{s,N}(\tau) U_N^\dagger |_{\tau=\tau(t)} = \frac{M\rho^2}{\Omega} O_N, \quad (14)$$

where

$$O_{s,N}(\tau) = -i\hbar \frac{\partial}{\partial \tau} + H_{s,N} \quad (15)$$

$$O_N(t) = -i\hbar \frac{\partial}{\partial t} + H_N. \quad (16)$$

This relation has been noticed for a specific case [17]. In Eq. (14), the unitary operator  $U_N$  is given as

$$U_N = \prod_{i=1}^N (\exp[\frac{i}{2\hbar} M \frac{\dot{\rho}}{\rho} x_i^2] \exp[-\frac{i}{4\hbar} \ln(\frac{\rho^2}{\Omega})(x_i p_i + p_i x_i)]) \quad (17)$$

$$= (\frac{\Omega}{\rho^2})^{N/4} \prod_{i=1}^N (\exp[\frac{i}{2\hbar} M \frac{\dot{\rho}}{\rho} x_i^2] \exp[-\frac{1}{2} \ln(\frac{\rho^2}{\Omega}) x_i \frac{\partial}{\partial x_i}]). \quad (18)$$

If  $\phi_s(x_1, x_2, \dots, x_N)$  is an eigenstate of Hamiltonian  $H_{s,N}$  with eigenvalue  $E$ , from Eqs. (14,18), the wave function  $\psi(t; x_1, x_2, \dots, x_N)$  satisfying  $O_N(t)\psi = 0$  is given as

$$\begin{aligned} & \psi(t; x_1, x_2, \dots, x_N) \\ &= e^{-iE\tau/\hbar} |_{\tau=\tau(t)} U_N \phi_s(x_1, x_2, \dots, x_N) \end{aligned} \quad (19)$$

$$\begin{aligned} &= (\frac{\Omega}{\rho^2})^{N/4} \left( \frac{u(t) - iv(t)}{\rho(t)} \right)^{E/\hbar} \left( \prod_{i=1}^N \exp[\frac{i}{2\hbar} M \frac{\dot{\rho}}{\rho} x_i^2] \right) \\ &\times \phi_s \left( \sqrt{\frac{\Omega}{\rho^2}} x_1, \sqrt{\frac{\Omega}{\rho^2}} x_2, \dots, \sqrt{\frac{\Omega}{\rho^2}} x_N \right). \end{aligned} \quad (20)$$

If  $V(x_1, x_2, \dots, x_N)$  is written in terms of the differences of positions of two particles, so that  $V$  satisfies Eq.(2), there is a unitary operator

$$U_f = e^{\frac{i}{\hbar} N \delta_f} \prod_{i=1}^N (\exp[\frac{i}{\hbar} M \dot{u}_f x_i] \exp[-\frac{i}{\hbar} u_f p_i]) \quad (21)$$

which does not change the  $O_N$  under a unitary transformation:

$$U_f O_N U_f^\dagger = O_N. \quad (22)$$

In Eq. (21),  $u_f$  is a linear combination of  $u(t), v(t)$ , and  $\delta_f$  is defined through the relation

$$\dot{\delta}_f = \frac{1}{2} M (w^2 u_f^2 - \dot{u}_f^2). \quad (23)$$

Therefore, in this case, a coherent wave function from  $\phi_s(x_1, x_2, \dots, x_N)$  is given as

$$\begin{aligned} \psi^f(t; x_1, x_2, \dots, x_N) \\ = e^{-iE\tau/\hbar} \big|_{\tau=\tau(t)} U_f U_N \phi_s(x_1, x_2, \dots, x_N) \end{aligned} \quad (24)$$

$$\begin{aligned} &= \left(\frac{\Omega}{\rho^2}\right)^{N/4} \left(\frac{u - iv}{\rho}\right)^{E/\hbar} e^{\frac{i}{\hbar} N \delta_f} \\ &\times \left(\prod_{i=1}^N \exp\left[\frac{i}{2\hbar} M \frac{\dot{\rho}}{\rho} (x_i - u_f)^2 + \frac{i}{\hbar} M \dot{u}_f x_i\right]\right) \\ &\times \phi_s\left(\sqrt{\frac{\Omega}{\rho^2}}(x_1 - u_f), \dots, \sqrt{\frac{\Omega}{\rho^2}}(x_N - u_f)\right). \end{aligned} \quad (25)$$

From the derivations through unitary transformations, it is manifest that

$$\int \prod_{i=1}^N dx_i \phi_s^* \phi_s = \int \prod_{i=1}^N dx_i \psi^* \psi = \int \prod_{i=1}^N dx_i \psi^f \psi^f. \quad (26)$$

Since the two unitary operators,  $U_N, U_f$  are invariant under the exchange of a pair of  $i$ -th and  $j$ -th particles, the wave functions  $\phi_s(x_1, x_2, \dots, x_N)$ ,  $\psi(t; x_1, x_2, \dots, x_N)$  and  $\psi^f(t; x_1, x_2, \dots, x_N)$  have the same symmetric property under the exchanges of particles. For the systems of identical particles, one of the quantities of interest which is independent of statistics [7,16,9] is the particle number density defined for  $\phi_s$  as

$$\sigma_s(x) = N \frac{\int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_N \phi_s^2(x, x_2, \dots, x_N)}{\int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N \phi_s^2(x_1, x_2, \dots, x_N)}. \quad (27)$$

From the Eq. (25), one can easily find the expression of the particle number density for  $\psi^f$  as

$$\sigma^f(x) = \frac{\sqrt{\Omega}}{\rho} \sigma_s\left(\frac{\sqrt{\Omega}}{\rho}(x - u_f)\right). \quad (28)$$

### III. APPLICATIONS

In this section, the general results of previous section will be explicitly applied for three cases. First, we will

consider the Sutherland model of Ref. [7]. We will find a general expression of a coherent state of the system (of time-dependent parameters) and will show that the expression reproduces the known state in the model [16]. Second, we will consider the model of three-body system [18,13]. Third, we will consider the Calogero model "in the Jacobi coordinate" [8] without the degree of freedom of center of mass.

#### A. Sutherland model

The system described by the Hamiltonian

$$H_{S,s} = \sum_{i=1}^N \left(\frac{p_i^2}{2} + \frac{x_i^2}{2}\right) + \sum_{i>j=1}^N \frac{\hbar^2 \lambda(\lambda-1)}{(x_i - x_j)^2} \quad (29)$$

has the (unnormalized) bosonic ground state [7]

$$\phi_S = \left(\prod_{j>i=1}^N |x_j - x_i|^\lambda\right) \prod_i^N e^{-x_i^2/2\hbar} \quad (30)$$

with the energy eigenvalue  $\hbar N[1 + \lambda(N-1)]/2$ . For the system described by the Hamiltonian

$$\begin{aligned} H_S &= \sum_{i=1}^N \left(\frac{p_i^2}{2M(t)} + M(t)w^2(t)\frac{x_i^2}{2}\right) \\ &+ \frac{1}{M(t)} \sum_{i>j=1}^N \frac{\hbar^2 \lambda(\lambda-1)}{(x_i - x_j)^2}, \end{aligned} \quad (31)$$

the wave function  $\psi_S^f$  satisfying

$$i\hbar \frac{\partial \psi_S^f}{\partial t} = H_S \psi_S^f \quad (32)$$

is given, from the results of previous section, as

$$\begin{aligned} \psi_S^f &= \left(\frac{u + iv}{\sqrt{\Omega}}\right)^{-N(1+\lambda(N-1))/2} e^{iN\delta_f/\hbar} \\ &\times \left(\prod_{i=1}^N \exp\left[\frac{i}{2\hbar} M \frac{\dot{\rho}}{\rho} (x_i - u_f)^2 + \frac{i}{\hbar} M \dot{u}_f x_i\right]\right) \end{aligned}$$

$$\times \left( \prod_{j>i=1}^N |x_j - x_i|^\lambda \right) \prod_{i=1}^N e^{-\Omega(x_i - u_f)^2 / 2\hbar\rho^2}. \quad (33)$$

In deriving Eq. (33), we make use of the fact that  $(u - iv)/\rho^2 = 1/(u + iv)$ . By making use of Eq. (28) and the results in Ref. [16,7,9], one can find the particle density for  $\psi_S^f$  is give as

$$\sigma_S^f(x) = \frac{\sqrt{2N\Omega}}{\pi\rho\sqrt{\lambda}} \sqrt{1 - \frac{\Omega}{2N\lambda\rho^2}(x - u_f)^2}. \quad (34)$$

By directly analyzing the Schrödinger equation (Eq. (32)), for the unit mass case, a coherent wave function is given in terms of a complex solution of classical equation of motion, and a hydrodynamic description was shown to hold exactly in the picture that the wave function provides [16]. If we choose  $u_f = 0$ , one can easily verify that, for the unit mass case, the  $\psi_S^f$  and  $\sigma_S^f$  reduce to the wave function and the density found by Sutherland [16], respectively.

As in the case of one-body harmonic oscillator [5], there are, in general, five free parameters in determining  $\psi_S^f$  or  $\sigma_S^f$ ; two of the parameters determine the motion of center of the particle number density, while the other three parameters determine the shape of the density function. To be explicit, we consider the particle density function of the case of unit mass and unit spring constant. In this case, two homogeneous solutions  $u(t), v(t)$  and the (fictitious) particular solution  $u_f(t)$  can be taken, without losing generality, as  $\cos(t + t_0)$ ,  $A \sin(t + \alpha + t_0)$  and  $B \cos(t + \beta)$ , respectively, with real constants  $t_0$ ,  $\beta$ , positive constants  $A$ ,  $B$ , and a real constant  $\alpha$  satisfying  $|\alpha| < \pi$ . Then the density function is written as

$$\frac{\sqrt{2NA \cos \alpha}}{\pi \tilde{\rho} \sqrt{\lambda}} \sqrt{1 - \frac{A \cos \alpha}{2N\lambda \tilde{\rho}^2} (x - B \cos(t + \beta))^2}, \quad (35)$$

with

$$\tilde{\rho} = \sqrt{\cos^2(t + t_0) + A^2 \sin^2(t + \alpha + t_0)}. \quad (36)$$

Due to the time-translational invariance, in this case, one of the parameters is simply related to the time shifting of the density functions.

## B. Three-body interaction model

As another example, we consider the three-body interaction model described by the Hamiltonian [18]:

$$H_{3body,s} = \sum_{i=1}^3 \left( \frac{p_i^2}{2} + \frac{x_i^2}{2} \right) + \sum_{i>j=1}^3 \frac{\hbar^2 \lambda (\lambda - 1)}{(x_i - x_j)^2} + 3 \sum_{i>j=1}^3 \frac{\hbar^2 \alpha (\alpha - 1)}{y_{ij}^2}, \quad (37)$$

where  $y_{ij}$  is defined as  $x_i + x_j - 2x_k$  ( $k \neq i$  and  $k \neq j$ ).

The (unnormalized) bosonic ground state is given as [18]

$$\phi_{3body} = \prod_{i>j=1}^3 (|x_i - x_j|^\lambda |y_{ij}|^\alpha) \prod_{i=1}^3 e^{-x_i^2/2\hbar} \quad (38)$$

with energy eigenvalue  $3\hbar(\frac{1}{2} + (\lambda + \alpha))$ . One can easily find that the potential of mutual interaction in  $H_{3body,s}$  is written in terms of the differences between positions of two particles. By applying the formulas in the previous section, one can find that the exact wave function for the system described by the Hamiltonian

$$H_{3body,s} = \sum_{i=1}^3 \left( \frac{p_i^2}{2M(t)} + M(t)w^2(t) \frac{x_i^2}{2} \right) + \frac{1}{M(t)} \sum_{i>j=1}^3 \frac{\hbar^2 \lambda (\lambda - 1)}{(x_i - x_j)^2} + \frac{1}{M(t)} \sum_{i>j=1}^3 \frac{\hbar^2 \alpha (\alpha - 1)}{y_{ij}^2}, \quad (39)$$

is given as

$$\begin{aligned} \psi_{3body}^f &= \left( \frac{u+iv}{\sqrt{\Omega}} \right)^{-3(\lambda+\alpha)-3/2} e^{3i\delta_f/\hbar} \\ &\times \prod_{i=1}^3 \exp \left[ \frac{i}{2\hbar} M \frac{\dot{\rho}}{\rho} (x_i - u_f)^2 + \frac{i}{\hbar} M \dot{u}_f x_i \right] \\ &\times \left( \prod_{i>j=1}^3 |x_i - x_j|^\lambda |y_{ij}|^\alpha \right) \prod_{i=1}^3 e^{-\Omega(x_i - u_f)^2 / 2\hbar\rho^2}. \end{aligned} \quad (40)$$

### C. Calogero model in Jacobi coordinates

The model described by the Hamiltonian

$$\tilde{H}_{C,s} = \sum_{i=1}^N \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i>j=1}^N (x_i - x_j)^2 + V_C \quad (41)$$

has been considered by Calogero [8], where

$$V_C = \sum_{i>j=1}^N \frac{\hbar^2 \lambda(\lambda-1)}{(x_i - x_j)^2}. \quad (42)$$

If one introduce the "Jacobi coordinates"

$$\begin{aligned} y_i &= \frac{1}{\sqrt{i(i+1)}} \left( \sum_{l=1}^i x_l - i x_{i+1} \right) \quad (i = 1, 2, \dots, N-1), \\ y_N &= \frac{1}{\sqrt{N}} \sum_{l=1}^N x_l, \end{aligned} \quad (43)$$

the Hamiltonian in Eq. (41) is written as

$$\tilde{H}_{C,s} = \frac{p_{y_N}^2}{2} + H_{C,s}, \quad (44)$$

where

$$H_{C,s} = \sum_{i=1}^{N-1} \left( \frac{p_{y_i}^2}{2} + \frac{y_i^2}{2} \right) + V_C. \quad (45)$$

It is easy to see that  $V_C$  does not depend on  $y_N$  and the Hamiltonian  $H_{C,s}$  describes a system of interacting  $N-1$  particles. In fact, Calogero analyzed the Hamiltonian system of  $H_{C,s}$ , and found the (unnormalized) wave functions

$$\phi_n^C(y_1, y_2, \dots, y_{N-1})$$

$$= \left( \prod_{i>j=1}^N (x_i - x_j)^\lambda \right) \exp \left( -\frac{1}{2\hbar} \sum_{i=1}^{N-1} y_i^2 \right) L_n^b \left( \frac{1}{\hbar} \sum_{i=1}^{N-1} y_i^2 \right) \quad (46)$$

satisfying

$$H_{C,s} \phi_n^C = E_n \phi_n^C \quad (n = 0, 1, 2, \dots), \quad (47)$$

where

$$b = \frac{1}{2}(N-3) + \frac{1}{2}\lambda N(N-1), \quad (48)$$

$$E_n = \hbar \left[ \frac{1}{2}(N-1) + \frac{1}{2}\lambda N(N-1) + 2n \right], \quad (49)$$

and  $L_n^b$  is the associated Laguerre polynomials defined through the equation

$$x \frac{d^2 L_n^b}{dx^2} + (b+1-x) \frac{dL_n^b}{dx} + n L_n^b(x) = 0. \quad (50)$$

If  $y_i$  is the space coordinate of the  $i$ -th particle, the Hamiltonian  $H_{C,s}$  does *not* describe the system of identical particles, as has been explicitly shown in the 3-body system [6].

Since the  $V_C$  can not be written in terms of  $y_i - y_j$ , only squeeze-type unitary transformation can be applied to give coherent states. For the system described by the Hamiltonian

$$H_C = \sum_{i=1}^{N-1} \left( \frac{p_{y_i}}{2M(t)} + M(t) \omega^2(t) \frac{y_i^2}{2} \right) + \frac{V_C}{M(t)}, \quad (51)$$

by making use of the unitary relation in Eq. (14), one may find that the wave functions satisfying  $i\hbar(\partial\psi_n^C/\partial t) = H_C\psi_n^C$  are given as

$$\begin{aligned} \psi_n^C &= \left( \frac{u+iv}{\sqrt{\Omega}} \right)^{-b-1} \left( \frac{u-iv}{\rho} \right)^{2n} \prod_{i>j=1}^N (x_i - x_j)^\lambda \\ &\times \exp \left[ -\frac{1}{2\hbar} \left( \frac{\Omega}{\rho^2} - iM \frac{\dot{\rho}}{\rho} \right) \sum_{i=1}^{N-1} y_i^2 \right] L_n^b \left( \frac{\Omega}{\hbar\rho^2} \sum_{i=1}^{N-1} y_i^2 \right). \end{aligned} \quad (52)$$

#### IV. A GENERALIZATION TO INCLUDE EXTERNAL FORCE

If  $V$  can be written in terms of the differences between positions of two particles, by modifying the  $U_f$ , one can find the unitary relation in different Hamiltonian systems, as in one-body harmonic oscillator [1]. By defining the  $U_F$  as

$$U_F = e^{\frac{i}{\hbar} N \delta_F} \prod_{i=1}^N (\exp[\frac{i}{\hbar} M \dot{x}_p x_i] \exp[-\frac{i}{\hbar} x_p p_i]) \quad (53)$$

where  $x_p$  and  $\delta_F$  are defined through the relations

$$\frac{d}{dt}(M \dot{x}_p) + M(t)w^2(t)x_p = F(t), \quad (54)$$

$$\dot{\delta}_F = \frac{1}{2}M(w^2 x_p^2 - \dot{x}_p^2), \quad (55)$$

one can find the relation

$$U_F O_N U_F^\dagger = -i\hbar \frac{\partial}{\partial t} + H_{N,F} \quad (56)$$

where

$$H_{N,F} = H_N - F \sum_{i=1}^N x_i. \quad (57)$$

From Eq. (56), one can find that the wave function

$$\begin{aligned} \psi^F &= \left(\frac{\Omega}{\rho^2}\right)^{N/4} \left(\frac{u - iv}{\rho}\right)^{E/\hbar} e^{\frac{i}{\hbar} N \delta_F} \\ &\times \left(\prod_{i=1}^N \exp\left[\frac{i}{2\hbar} M \frac{\dot{\rho}}{\rho} (x_i - x_p)^2 + \frac{i}{\hbar} M \dot{x}_p x_i\right]\right) \\ &\times \phi_s\left(\frac{\sqrt{\Omega}}{\rho}(x_1 - x_p), \dots, \frac{\sqrt{\Omega}}{\rho}(x_N - x_p)\right) \end{aligned} \quad (58)$$

satisfies the Schrödinger equation

$$i\hbar \frac{\partial \psi^F}{\partial t} = H_{N,F} \psi^F. \quad (59)$$

#### V. SUMMARY AND DISCUSSIONS

It has been shown that the unitary relations in one-body harmonic oscillator systems can be extended to give the unitary relations in some of the Calogero-Sutherland models. These unitary relations can be used not only in finding coherent states from the given stationary states in a system, but also in finding exact wave functions of the Hamiltonian systems of time-dependent parameters from those of time-independent Hamiltonian systems. If the potential of mutual interactions can be written in terms of the differences between positions of two particles, we have also shown that the wave functions of the system with external force can be found from those of the the same system without the external force. The list of applications given in this paper is *not* exhaustive.

The unitary relations can be formally extended to the case of identical  $N$ -body free particles interacting through the mutual interaction potential  $V$ ; however, in this case, the  $\rho(t)$  diverges as  $t^2$  goes to infinity. Even for the case of identical  $N$ -body harmonic oscillators (interacting through the  $V$ ),  $\rho(t)$  could be unbounded in general, as analyzed in detail for the cases of periodic mass and frequencies [19,16].

The system of identical  $N$ -body free particles interacting through a potential  $\frac{\hbar\lambda(\lambda-1)\pi^2}{L^2} \sum_{i>j}^N \frac{1}{\sin^2[\pi(x_i - x_j)/L\sqrt{\hbar}]}$  with a constant  $L$  [11], has recently been of great interest [10,14,15], while the potential is not of homogeneous degree -2. Since the potential satisfies Eq. (2),

for the unit mass case, if one choose  $u_f$  as  $at + b$  with constants  $a, b$ , one can show that the Schrödinger equation is invariant under a unitary transformation, as in Eq. (22) (with  $w = 0$ ). For this case, the (unnorm-  
malized) wave function of the ground state is given as  $\psi_0 = \prod_{i>j} (\sin \frac{\pi(x_i - x_j)}{L\sqrt{\hbar}})^\lambda$ . By applying the unitary transformation of  $U_f$  in Eq. (21) to the  $\psi_0$ , one can obtain the wave function  $\psi_a = [\prod_{i=1}^N \exp(iax_i/\hbar)]\psi_0$ , up to a purely time-dependent phase.  $\psi_a$  is an eigenstate of the Hamiltonian. The  $\psi_a$  has been discussed as an excited state which may be obtained by implementing a Galilei boost to  $\psi_0$  [20], while the derivation in this paper clearly supports this interpretation.

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